by Jensen's formula. Hence, if $\liminf_{r \to \infty} r^{-\rho} n(r) = \beta$, we have for any $\varepsilon > 0$ and $r > r_0$, $(\beta - \varepsilon) r^{\rho} < n(r)$. Using this inequality in (2.1) the result of (1.9) follows.

Again, if $\liminf_{r \to \infty} n(r)/r \log r > 1$, we have for any $\varepsilon > 0$ and $r > r_0 > 0$

$$(1 - \varepsilon) r \log r < n(r)$$

and using this in (2.1), we have

$$\log I_\delta(r) > \delta (1 - \varepsilon) \int_{r_0}^{r} \log t \, dt + \delta \log |f(0)|$$

Therefore,

$$\liminf_{r \to \infty} \frac{\log I_\delta(r)}{r \log r} \geq \delta$$

Further, if $|f(0)| = 1$, then from (1.3) and (2.1), we obtain

$$N(r) \leq \log \{ I_\delta(r) \}^{1/\delta} \leq \log M(r).$$

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THE AUTOMORPHISMS OF $PS_4^+(Q)$

por

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The automorphisms of the group of similitudes and some related groups have been studied in [7]. However, for the case of a 4-dimensional vector space the only group which was considered there was the group of similitudes. In the present paper we study the projective group of proper similitudes $PS_4^+(Q)$ of a 4-dimensional vector space with a non-degenerate quadratic form Q. First we find groups isomorphic to $PS_4^+(Q)$ by using the Clifford algebra C (Q) defined by Q or rather its subalgebra of even elements $C^+(Q)$. Then, in section 2, we determine the automorphisms of $PS_4^+(Q)$.

We will use some results in [6] as well as the well-known results about the structure and some of the properties of $C^+(Q)$ when Q is a non-degenerate quadratic form over a vector space of dimension 3 or 4. For this we refer the reader to [1, Chap. V, § 6], [2, II, § 9] or [3, I, § 5].

1. Let M be a vector space of dimension 4 over the (commutative) field K and Q a non-degenerate quadratic form over Q. It is always assumed that we deal with fields of *characteristic different from 2*.

To describe the structure of $C^+(Q)$ we have to consider two different cases.

Case 1. Q has a non-square discriminant. Then $C^+(Q)$ is a generalized quaternion algebra over the field $F = K(\sqrt{\Delta})$, where

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Δ is any element in the quadratic class of the discriminant of Q. Thus $C^+(Q)$ is either isomorphic to the algebra F_2 of 2×2 matrices with entries in F or it is a division algebra. $C^+(Q)$ is isomorphic to F_2 if, and only if, Q has index 1 (see [2, II, § 9], BI).

Case 2. Q has square discriminant. Then $C^+(Q)$ is the direct sum of two isomorphic quaternion algebras over K, say $B \oplus B$.

The main involutory anti-automorphism of $C^+(Q)$ will be denoted by J. In case 1, J is the anti-automorphism of $C^+(Q)$ which leaves the elements of F invariant and has the property that $u u^J \in F$ for every $u \in C^+(Q)$. In case 2, J induces on each summand the anti-automorphism which leaves the elements of K invariant and has the property $u u^J \in K$ for every $u \in B$. For such anti-automorphism the elements of the quaternion algebra satisfying $u^J = -u$ can be characterized as those elements not in the center whose square belongs to the center.

When we are dealing with an algebra B with identity e over a field K, we will talk of an automorphism Σ of B as a ring to mean that such automorphism may or may not take the scalar multiples of the identity into themselves. If we are in the first case the automorphism Σ induces an automorphism, say σ, on the elements $K.e$ and we will say that Σ is a semi-automorphism of B as a K-algebra, in particular, if σ is the identity we say that Σ is an automorphism of the K-algebra.

If no field is explicitly mentioned, $C^+(Q)$ is considered as a K-algebra, where K is the base field of the vector space M.

When B is an algebra or a field we denote by B^* the multiplicative group of invertible elements of B.

THEOREM 1.—Let $P S^+(Q)$ be the projective group of proper similitudes of M with respect to Q. Then $P S^+(Q)$ is isomorphic to the group of invertible elements of $C^+(Q)$ modulo its center.

PROOF.—An element of $C^+(Q)$ satisfies $u^J = -u$ if, and only if, $u \in M[_2] \subset C^+(Q)$ (see [6,I] for the definition of $M[_2]$). In case 1 as well as in case 2 any automorphism of $C^+(Q)$ as a ring takes elements satisfying $u^J = -u$ into elements with the same property; in other words, any automorphism of $C^+(Q)$ as a ring leaves invariant the subpace $M[_2]$. Therefore, by [6, Thm. 4 and its Corollary 1], the group $P S^+(Q)$ is isomorphic to the group of inner automorphisms of $C^+(Q)$. It is clear that the latter group is isomorphic to the group defined in the statement of the theorem.

COROLLARY.—Let $P S^+(Q)$ be as in the theorem. Then:

1) If Q has non-square discriminant $P S^+(Q) \cong O_3^+(Q')$, the group of rotations of a vector space N of dimension 3 over the field $F = K(\sqrt{\Delta})$ with respect to the quadratic form Q'. Q' is obtained in the following way.

Let M_1 be any non-isotropic subspace of M of dimension 3 and Q_1 the restriction of Q to M_1. Then Q' is the extension of Q_1 to the space $N = F \oplus_K M_1$.

If index of Q = 1, $P S^+(Q) \cong P G L_2(F)$ the projective linear group. If index of Q = 0, $P S^+(Q) \cong B^*/F^*$, where B is a quaternion division algebra over F.

2) If Q has square discriminant,

$$P S^+(Q) \cong Q_3^+(Q') \times O_3^+(Q'),$$

the direct product of two copies of the group of rotations of any non-isotropic 3-dimensional subspace M_1 of M with respect to the restriction Q' of Q to M_1.

When index of Q = 2,

$$P S^+(Q) \cong P G L_2(K) \times P G L_2(K)$$

and if index of Q = 0,

$$P S^+(Q) \cong B^*/K^* \times B^*/K^*,$$

B being a quaternion division algebra over K.

PROOF.—Let x_1, x_2, x_3 be an orthogonal basis of a non-isotropic subspace M_1 of M and choose x_4 orthogonal to M_1. Then the elements $1, x_1 x_2, x_1 x_3, x_2 x_3$ form a basis of $C^+(Q)$ with respect to $K + K x_1 x_2 x_3 x_4$.

1) If Q has non-square discriminant,

$$K + K x_1 x_2 x_3 x_4 \cong K(\sqrt{\Delta}) = F$$

and $C^+(Q)$ is a quaternion algebra over F. It is well-known that $(C^+(Q))^*/F^*$ is isomorphic to the groups of rotations of the 3-dimensional vector space N over F with respect to the quadratic form Q′ described in the corollary. If index of $O = 1$, $C^+(Q)) \cong F_2$ and, if index of $Q = 0$, $C^+(Q)$ is a division ring.

2) If Q has square discriminant $K + x_1 x_2 x_3 x_4 \cong K \oplus K$, and $C^+(Q) \cong B \oplus B$, where B is a quaternion algebra over K isomorphic to the quaternion algebra with basis 1, $x_1 x_2$, $x_1 x_3$, $x_2 x_3$. Hence $B \cong C^+(Q')$, where Q′ is the restriction of Q to M_1, and

$$P\,S_4^+(Q) \cong (C^+(Q))^*/(K+K)^* \cong B^*/K^* \times B^+/K^* \cong$$
$$\cong O_3^+(Q') \times O_3^+(Q')$$

Since the discriminant of Q is a square, either index of $Q = 2$ or index of $Q = 0$. If index of $Q = 0$, then index of $Q' = 0$ and B is a quaternion division algebra, and index of $Q = 2$ implies index of $Q' = 1$, for, if P is a 2-dimensional totally isotropic subspace of M,

$$\dim(M_1 \cap P) = \dim M_1 + \dim P - \dim(M_1 + P) =$$
$$= 3 + 2 - \dim(M_1 + P) \geqslant 1$$

and, since Q′ is non-degenerate we get index of $Q' = 1$. Thus, if index of $Q = 2$,

$$P\,S^+(Q) \cong P\,G\,L_2(K) \times P\,G\,L_2(K).$$

2. To determine the automorphisms of $P\,S^+(Q)$ we have to find the automorphisms of the group B^*/K^* where B is a quaternion algebra over K. If $B \cong K_2$, then $B^*/K^* \cong P\,G\,L_2(K)$ and the automorphisms of this group have been determined by L. K. Hua (see [4, Thm 3]; notice that when K is commutative and $A \in K_2$, $(A')^{-1} = (\det. A)^{-1} C A C^{-1}$ where $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; hence the automorphisms of $P\,G\,L_2(K)$ are induced by the semi-automorphism of the K-algebra K_2, two different semi-automorphisms induce different automorphisms in $P\,G\,L_2(K)$).

We are left with the case when B is a quaternion division algebra. In this case we can identify the elements of the group B^*/K^* with the points of the projective geometry P (B), if we consider B as a vector space over K. Then the group multiplication defines a binary operation among the points of P (B).

The quaternion algebra B with the automorphism J defined above is the composition algebra of the vector space B with respect to the quadratic form $Q(u) = u\,u^J$ (see [5]). We will say that two elements of B are orthogonal vectors, if they are orthogonal with respect to the symmetric bilinear form associated with Q, or, what is the same, if $x y^J + y x^J = 0$. The elements orthogonal to the identity element e of the algebra B form a vector space of dimension 3 which we denote by B_0; hence

$$B_0 = \{ u \mid u^J = -u, u \in B \}.$$

Lemma 1.—Let x and y be non-zero vectors in B. Then x and y are orthogonal if, and only if, the coset of $x^{-1} y$ in B^*/K^* is an element of order exactly 2.

Proof.—Since

$$x x^J = Q(x),\ x^{-1} = Q(x)^{-1} x^J.$$

Now, to say that the coset of $x^{-1} y$ has order exactly 2 is equivalent to saying that $x^J\ y \in B_0$, that is,

$$(x^J y)^J = -x^J y \text{ or } y^J x + x^J y = 0,$$

which is equivalent to the orthogonality of x and y.

Theorem 2.—Let B be a quaternion algebra over K. The group of automorphisms of B^*/K^* is isomorphic to the group of semi-automorphisms of the K-algebra B.

Proof.—We have already mentioned that when $B \cong K_2$ the result is due to L. K. Hua. Let us assume then that B is a quaternion division algebra over K. We want to show that any automorphism of B^*/K^* defines a collineation of P (B). First we notice that any automorphism of B^*/K^* transforms the subspace $P(B_0)$ onto itself, because the elements belonging to B_0^*/K^* are the only ones of order exactly 2. In fact, any automorphism φ of B^*/K^* induces a collineation in $P(B_0)$. For let X_1, X_2, X_3 be three different collinear points in $P(B_0)$ and let $X^{\varphi}_1, X^{\varphi}_2, X^{\varphi}_3$ be their images under φ. If x_1, x_2, x_3 are three vectors in the cosets X_1, X_2, X_3, respectively, x_1, x_2, x_3 span a 2-dimensional vector space L and consequently there exists a vector $0 \neq z \in B_0$ orthogonal to L. Let x'_1, x'_2, x'_3 be three vectors in the cosets $X^{\varphi}_1, X^{\varphi}_2, X^{\varphi}_3$, respectively, if z' is in the coset Z^{φ}, $z' \in B_0$ and lemma 1 implies that z' must be orthogonal to x'_1, x'_2, x'_3. This is only possible if x'_1, x'_2, x'_3 are linearly dependent, hence φ induces a collineation in $P(B_0)$.

Next, let us assume that C_1, C_2, C_3 are three collinear points of $P(B)$ not all in $P(B_0)$. Then at least two of them are not in $P(B_0)$ and we can choose vectors of the form $e + x, e + y, u$ in the cosets C_1, C_2, C_3, with $x, y \in B_0$. Let $0 \neq z \in B_0$ be a vector orthogonal to x and y. Then the points D_1, D_2, D_3 defined by $(e + x) z$, $(e + y) z$, $u z$ are also collinear and belong to $P(B_0)$. Now, if $e + x'$, $e + y'$, u' are vectors in $C^{\varphi}_1, C^{\varphi}_2, C^{\varphi}_3$ and z' is in the image of the coset of z, the points $D^{\varphi}_1, D^{\varphi}_2, D^{\varphi}_3$ will be three collinear points of $P(B_0)$ defined by the vectors $(e + x') z'$, $(e + y') z'$, $u' z'$, which implies that $C^{\varphi}_1, C^{\varphi}_2, C^{\varphi}_3$ are collinear.

We have proved then that φ defines a collineation in $P(B)$, hence by the fundamental theorem of projective geometry we know that φ is induced by a semi-linear transformation S' of the vector space B and that such transformation is uniquely defined up to a scalar. Since φ leaves invariant the point defined by e, $e S' = \alpha e$ and by multiplying by a scalar we can choose a semi-linear transformation S such that $e S = e$. But φ is an automorphism of B^*/K^*, therefore the coset of $x^{-1} S$ is the inverse of the coset of $x S$. Now, if the coset of $x^{-1} y$ has order two so does the coset of $(x S)^{-1} (y S)$ which implies that S takes orthogonal vectors into orthogonal vectors, that is, S is a semi-similitude with respect to the quadratic form Q. Since $e S = e$, S has ratio 1 and we have to prove now that our S defines a semi-automorphism of the composition algebra B.

Assume then that T is a semi-similitude of B with respect to Q relative to the automorphism σ of K and that T leaves invariant the vector e and induces an automorphism in B^*/K^*. Let

x_1, x_2, x_3 be an orthogonal basis of B_0 such that $x_1 x_2 = x_3$, then either $(x_1 T)(x_2 T) = x_3 T$ or $(x_1 T)(x_2 T) = -x_3 T$. If the latter case occurs we get

$$x_1 (x_1 + x_3) = x_1 (x_1 + x_1 x_2) = -Q(x_1)(e + x_2)$$

but

$$x_1 T (x_1 T + x_3 T) = x_1 T (x_1 T) - (x_1 T)(x_2 T) = \\ = -Q(x_1 T)(e - x_2 T),$$

which shows that T does not induce an automorphism of B^*/K^*. Hence $(x_1 T)(x_2 T) = x_3 T$ and it is easily seen that T defines a semi-automorphism of B. Thus we have proved that the automorphisms of B^*/K^* are induced by the semi-automorphisms of the K-algebra B and it is clear now that the correspondence between the semi-automorphisms of B and the automorphisms that they induce in B^*/K^* is an isomorphism.

Corollary.—The group of automorphisms of the group of rotations of a 3-dimensional vector space over a field K with a non-degenerate quadratic form Q is isomorphic to $P\Gamma S_3(Q)$, the projective group of semi-similitudes with respect to Q.

Proof.—It is well-known that $O_3^+(Q) \cong (C^+(Q))^*/K^*$ and that $C^+(Q)$ is a quaternion algebra over K. By the preceding theorem we know that the group of automorphisms of $O_3^+(Q)$ is isomorphic to the group of semi-automorphisms of the algebra $C^+(Q)$. Since in the present case any semi-automorphism of $C^+(Q)$ leaves invariant the subspace $M_{[2]}$, by [6, Corollary 1 to Thm. 4] the group of semi-automorphisms of $C^+(Q)$ is isomorphic to $P\Gamma S_3(Q)$.

Combining this Corollary with part 1) of the Corollary of Theorem 1 we get.

Theorem 3 (I).—Let M be a 4-dimensional vector space over K and Q a non-degenerate quadratic form on M with non-square discriminant. Then the group of automorphims of $PS_4^+(Q)$ is isomorphic to the proyective group of semi-similitudes $P\Gamma S_3(Q')$, where Q' is the quadratic form defined in part 1 of the Corollary of Thm. 2.

It was shown in [7] that if M is a vector space over a fie'd K of characteristic $\neq 2$, dim $M > 4$ but $\neq 8$ and Q a non-degenerate quadratic form over M, then the automorphisms of the projective group of proper similitudes $P\,S^+(Q)$ can be written in the form

$$\overline{X}^{\varphi} = \overline{Y^{-1} X Y}$$

where $\overline{Z}$ stands for the coset of $Z \in S^+(Q)$ in $P\,S^+(Q)$ and Y is an element of the group of semi-similitudes $\Gamma S(Q)$ defined up to a scalar, that is, φ is defined by an unique element of $P\,\Gamma\,S^+(Q)$. We will refer to automorphisms of this form as standard automorphisms.

COROLLARY.—Let Q and Q′ be as in Theorem 3. If the element $\overline{T} \in P\,\Gamma\,S_3(Q')$ is defined by the semi-similitude $T \in \Gamma\,S_3(Q')$ relative to the automorphism τ of F, then $\overline{T}$ defines a standard automorphism of $P\,S_4^+(Q)$ if, and only if, the automorphism τ leaves invariant the subfield K.

PROOF.—Any automorphism φ of $C^+(Q)$ as a ring induces an automorphism $\overline{\varphi}$ in $P\,S_4^+(Q) \cong (C^+(Q))^*/F^*$, but φ, and hence also $\overline{\varphi}$, is associated to an element of $P\,\Gamma\,S_4^+(Q)$ if, and only if, φ is a semi-automorphism of the K-algebra $C^+(Q)$ (cf. [6, Thm. 4]).

Now $\overline{T}$ defines a semi-automorphism of the F-algebra $C^+(Q)$, but $\overline{T}$ is a semi-automorphism of $C^+(Q')$ as a K-algebra if, and only if, τ leaves invariant the subfield K. Since as K-algebras $C^+(Q') \cong C^+(Q)$, the corollary follows.

We give now an example of a group $P\,S_4^+(Q)$ with non-standard automorphisms. Let M be the 4-dimensional space over the field of real numbers R and with a quadratic form Q of index 1. Then $P\,S_4^+(Q) \cong \Omega_4(Q) \cong P\,S\,L_2(C)$, where $\Omega_4(Q)$ stands for the commutator group of $O^+(Q)$ and C for the complex numbers. The semi-automorphisms of C_2 relative to the non-continuous automorphisms of the complex numbers induce non-standard automorphisms of the Lorentz group $\Omega_4(Q)$.

Our next aim is to study the automorphisms of $P\,S^+(Q)$ when the quadratic form Q has a square discriminant.

LEMMA 2.—If B is a quaternion algebra over K the group B^*/K^* is generated by the elements of order 2.

PROOF.—Let $z \in B^*$ define a coset of B^*/K^* not of order 2. Then such coset contains an element of the form $e + x$ where $0 \neq x \in B_0$. If y is a non-isotropic vector of B_0 orthogonal to x $(e + x)\,y = u \in B_0$, hence the coset of z is the product of the cosets of u and y.

LEMMA 3.—Let $G = B^*/K^* \times B^*/K^*$, $G_1 = B^*/K^* \times 1$ and $G_2 = 1 \times B^*/K^*$, where 1 denotes the identity element of B^*/K^*. Then, if K has more than 3 elements, any automorphism of G either takes G_1 and G_2 into themselves or permutes them.

PROOF.—We are going to assume that B is a quaternion division algebra since when $B \cong K_2$, $K \neq GF(3)$ the result is well-known being a consequence of the fact that the commutator group of G is the direct product of two simple groups, namely, $P\,S\,L_2(K) \times P\,S\,L_2(K)$ (see [2, p. 96]).

Obviously $(h_1 \times h_2)^{\varphi} = h_2 \times h_1$ is an automorphism of G, Since G_1 is generated by the elements of order 2, we only need prove that if Ψ is an automorphism of G either Ψ takes all the elements of order 2 of G_1 into elements of G_1 or Ψ_{φ} does.

Let $h = h_1 \times b_2 \neq 1 \times 1$, $h^2 = 1 \times 1$ and define

$$S_h = \{z \mid (z\,h)^2 = 1 \times 1,\ z^2 = 1 \times 1,\ z \in G\}$$

Then, if $C(S_h)$ is the centralizer of S_h, it is easily seen that

$$C(S_h) = \{1 \times 1,\ 1 \times h_2\} \quad \text{if} \quad h_1 = 1$$
$$C(S_h) = \{1 \times 1,\ h_1 \times 1\} \quad \text{if} \quad h_2 = 1$$

and $C(S_h) = \{1 \times 1, 1 \times h_2, h_1 \times 1, h_1 \times h_2\}$ if $h_1, h_2 \neq 1$.

For, if $h_1 \neq 1$, let e, x_1, x_2, x_3 be an orthogonal basis of the composition algebra B such that x_1 belongs to the coset h_1. Then the elements of orden 2 of B^*/K^* which commute with h_1 are the cosets of vectors of the form $\alpha x_2 + \beta x_3 \neq 0$ and 1 and h_1 are the only cosets which commute with the cosets of all such elements. If $h_1 = 1$, all elements commute with h_1 and 1 is the only element which commutes with all the elements of order 2.

Now, if $h = h_1 \times 1$ and $g = g_1 \times g_2$, $g_1, g_2 \neq 1$, the subgroups $C(S_h)$ and $C(S_g)$ are not isomorphic. Hence an automorphism Ψ of G must take h either into an element of the form $h'_1 \times 1$ or into an element $1 \times h'_2$. By replacing, if necessary, Ψ by Ψ_φ, we can assume that $(h_1 \times 1)^\Psi = h'_1 \times 1$. Then, if $f = f_1 \times 1$ is another element of order 2, $(f_1 \times 1)^\Psi = f'_1 \times 1$, for from $(f_1 \times 1)^\Psi = 1 \times f'_2$ follows, $(h f)^\Psi = h'_1 \times f'_2$ and $(h f)^2 = ((h'_1 \times f'_2)^2)^{\Psi^{-1}} = 1 \times 1$, that is, the element of order 2 $h f = h_1 f_2 \times 1$ is mapped into $h'_1 \times f'_2$ with $h'_1, f'_2 \neq 1$, and we have just shown that this can not hapepn.

COROLLARY.—If the field K has more than three elements and B is a quaternion algebra over K, the automorphisms of the group

$$E^*/K^* \times B^*/K^* \cong (B + B)^*/(K + K)^*$$

are induced by the automorphisms of the ring $B \oplus B$.

THEOREM 3 (II).—Let M be a 4-dimensional vector space over K, a field with more than 3 elements, and let Q be a non-degenerate quadratic form on M. If Q has a square discriminant,

$$C^+(Q) \cong B \oplus B$$

where B is a quaternion algebra over K and the automorphisms of $P\,S^+(Q) \cong B^*/K^* \times B^*/K^*$ are induced by the ring automorphisms of $C^+(Q)$. More precisely, let $\overline{\Sigma}_1, \overline{\Sigma}_2$ be any two automorphisms of B^*/K^* induced by the semi-automorphisms Σ_1 and Σ_2 of B relative to the automorphisms σ_1 and σ_2 of K, then the automorphims Ψ of $P\,S^+(Q)$ is either of the form

$$(g_1 \times g_2)^\Psi = g_1^{\overline{\Sigma}_1} \times g_2^{\overline{\Sigma}_2} \text{ or } (g_1 \times g_2)^\Psi = g_2^{\overline{\Sigma}_2} \times g_1^{\overline{\Sigma}_1}$$

Such a Ψ is an standard automorphism of $P\,S^+(Q)$ if, and only if, $\sigma_1 = \sigma_2$.

PROOF.—The only part which needs proof is the statement about standard automorphisms. This follows from [6, Thm. 4] as in the proof of the Corollary of Theorem 3 (I) taking into account that now the scalar multiples of the identity element of the algebra $C^+(Q)$ are of the form $\alpha e \oplus \alpha e$, $\alpha \in K$.

REMARK.—Let M be a 4-dimensional vector space over K and Q a non-degenerate quadratic form on M. Asume that Q has a square discriminant, and, by multiplying Q, if necessary, by a

scalar, we can also assume that there exists a vector $e \in M$ such that $Q(e) = 1$. Then we can define a bilinear multiplication in M so that it becomes a composition algebra with respect to Q. Using this multilpication any proper similitude S of M with respect to Q can be represented by two invertible elements (q_1, q_2) of M in the following way $x\,S = q_1 \times q_2$. Two pairs (p_1, p_2), (q_1, q_2) define the same similitude if and only if $(q_1, q_2) = (\lambda p_1, \lambda^{-1} p_2)$ and (q_1, q_2) defines a rotation of and only if $Q(q_2) = Q(q_1)^{-1}$. This representation allows us to extend the method used in [2, p. 96] to study $O^+(Q)$ when Q has index two to the case of any non-degenerate quadratic form with square discriminant (assuming that K has more than 3 elements). It follows immediately from lemma 3 above that in that case any automorphism of $S^+(Q)$ takes any orthogonal involution of $O^+(Q)$ into another orthogonal involution, consequently it takes $O^+(Q)$ into itself. Thus the automorphisms of $S^+(Q)$ are extensions of automorphisms of $O^+(Q)$.

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