THE AUTOMORPHISMS OF $U_n^+(k,f)$ AND $P\,U_n^+(k,f)$

by

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RESUMEN

J. Dieudonné ha determinado los automorfismos de $U_n^+(k,f)$ —grupo de las transformaciones unitarias de una forma hermética no degenerada f sobre un espacio vectorial de dimensión n sobre el cuerpo k con respecto a la involución * de k— admitiendo que k tiene característica distinta de 2, $n \geqslant 3$, y f tiene un índice mayor que cero. Para las formas de índice cero demuestra que se puede aplicar un método (de Mackey-Rickart) cuando n sea impar. En el presente trabajo se amplia la aplicación de este método a determinadas involuciones del grupo unitario de f cuando n es par y mayor que 6 y para cualquier índice de f. Se generaliza finalmente un teorema de Dieudonné.

Let M be a vector space of dimension n over the (commutative) field k and let * be an involution of k different from the identity. If f is a non-degenerate hermitian form over M with respect to the involution * of k, $U_n^+(k,f)$ denotes the group of unitarian transformations of f with determinant 1. Sometimes the subindex n will be omitted and if f has an orthonormal basis we write simply $U_n^+(k)$. The center Z of the group $U_n^+(k,f)$ consists of the transformations which coincide with scalar multiplication by an element $\lambda \in k$ such that $\lambda\lambda^* = 1$ and $\lambda^n = 1$. The corresponding projective group $U^+(k,f)/Z$ is denoted by $P\,U^+(k,f)$. In [2, XVII, Thm. 26] J. Dieudonné determined the automorphisms of $U_n^+(k,f)$ assuming that k has characteristic $\neq 2$, $n \geqslant 3$ and f is a hermitian form of index greater than zero. In [3, Chapter IV, § 5] Dieudonné points out that when n is odd using the method of Mackey-Rickart it can be proved that the statement of Theorem 26 applies also to forms of index 0. We will show in section 1 below that when $n > 6$ is even the

method of Mackey-Rickart can also be applied to characterize the involutions of type $(2, n-2)$ and $(n-2, 2)$ of $U_n^+(k,f)$ without any restriction on the index of f. Once these involutions have been distinguished from the $(p, n-p)$ involutions with p and $n-p$ greater than 2, the second part of Dieudonne's argument can be applied so that Theorem 25 can be generalized in the following form.

For $n \geqslant 3$, $n \neq 4$, *with the possible exception of the two groups* $U_3^+(F_9)$ *and* $U_3^+(F_{25})$, *every automorphism* φ *of the group* $U_n^+(k, f)$, *when* k *is a field of characteristic* $\neq 2$, *can be written in the form*

$$S^{\varphi}\, U\, S\, U^{-1} \chi(U)$$

U *being a unitarian semi-similtude relative to an automorphism of* k *which commutes with the involution* * *and* χ *is a representation of* $U_n^+(k, f)$ *in its center.*

As to the group $P\,U_n^+(k,f)$ Dieudonné has found, [2, XIX and XX], its automorphisms assuming that $n \geqslant 3$, $n \neq 4$ and k is a finite field. We will show below that Dieudonné's result holds for any field of characteristic $\neq 2$. Actually our proof excludes the cases when k has 9 or 25 elements but these are covered by Dieudonné. Thus Theorem 28 of [2] can be stated as follows:

For $n \geqslant 3$, $n \neq 4$, *with the possible exception of* $P\,U_3(F_9)$ *and* $P\,U_3(F_{25})$, *every automorphism* φ *of* $P\,U_n^+(k, f)$, k *a field of characteristic* $\neq 2$, *is induced by an automorphism of* $U^+(k, f)$.

1. We will consider M as a left vector space and will write the transformation to the right. The elements of the subfield $k_0 = \{\alpha \mid \alpha = \alpha^*, \alpha \in k\}$ will denoted by lower case latin letters and we always assume that k is a field of characteristic $\neq 2$. Then it is well-known that $k = k_0(\theta)$, where $\theta^2 = r$ and $\theta^* = -\theta$. The transformation which takes any $x \in M$ into αx, will be denoted by α_L.

As usual we call an element of order at most 2 of the group $U_n(k,f)$ a unitarian involution. Any unitarian involution T gives a decomposition of M in a direct sum of two non-isotropic subspaces, $M = M^+ \oplus M^-$, such that T leaves invariant the elements of the plus-space M^+ and $-T$ leaves invariant the elements of the minus-space M^-. If M^+ has dimension p we say that T is a $(p, n-p)$ involution; clearly, if $T \in U_n^+(k, f)$, $n-p$ is even. The involution T is completely defined if we know M^+ or M^- since each one of these subspaces is the orthogonal complement of the other. If R is a unitarian transformation which commutes with T, then R induces unitarian transformations in M^+ and M^- and $R \in U_n^+(k, f)$ if and only if the product of the determinants of these two induced transformations is 1. If R anticommutes with T, that is, $RT = TR$, then $M^+ R = M^-$ and $M^- R = M^+$, which implies that M^+ and M^- have the same dimension.

When $\{A_1, A_2, \ldots, A_r\}$ is a subset of elements of a group G we denote by $C_G(A_1, A_2, \ldots, A_r)$ the centraliser of the subset in G, that is, the group consisting of the elements of G wich commute with every element of the given subset. The centralizer of this centralizer is called the double centralizer.

Lemma 1. If $\dim M = n = 2m$, $m > 3$, in the group $U_n^+(k, f)$ the $(2, n-2)$ and $(n-2, 2)$ involutions have different group-theoretic properties that the $(p, n-p)$ involutions with p and $n-p$ both greater than 2.

Proof. Let T be a $(2, n-2)$ involution and $R \neq 1_L$, T any involution in $U^+(k, f)$ which commutes with T. If M^+ and M^- are the plus- and minus-spaces of T and N^+ and N^- those of R, then

$$M = M^+ \cap N^+ \oplus M^+ \cap N^- \oplus M^- \cap N^+ \oplus M^- \cap N^-.$$

If one of these 4 spaces is zero the double centralizer

$$C_{U^+}\left(C_{U^+}(T, R)\right)$$

contains only 8 involutions. If none of these 4 spaces is zero then $M^+ \cap N^+$ and $M^+ \cap N^-$ are both 1-dimensional and in this case too $C_{U^+}(C_{U^+}(T, R))$ contains only 8 involutions. It is clear that we get the same results if T is a $(n-2, 2)$ involution.

On the other hand, if $T \in U^+(k, f)$ is a $(p, n-p)$ involution with p and $n-p$ both greater than 2, we can take non-isotropic 2-dimensional subspaces W_1 and W_2 in the plus- and minus-spaces of T. Then the involution R which has $W_1 + W_2$ as plus-space commutes with T and there exist 16 involutions in the double centralizer of $\{T, R\}$ in $U_n^+(k, f)$.

Hence if T is a $(2, n-2)$ or $(n-2, 2)$ involution the double centralizer of $\{T, R\}$, where $R \neq 1_L$, T is any involution of

$U^+(k, f)$ commuting with T, contains 8 involutions and this is not true for the $(p, n-p)$ involution with p and $n-p$ both greater than 2. This proves the lemma.

2. We turn now to $P\,U^+(k, f)$. If $S \in U^+(k, f)$ then $\overline{S}$ denotes the coset of S in $P\,U^+(k, f)$ and S is an involution of $P\,U^+(k, f)$ if and only if $S^2 \in Z$. When $S^2 = \alpha^2_L$ with $\alpha_L \in Z$, since $\overline{S} = \overline{S\,\alpha_L^{-1}}$ the coset S contains $S\,\alpha_L^{-1}$ which is a $(p, n-p)$ unitarian involution with $n-p$ even and we will say that $\overline{S}$ is a $(p, n-p)$ coset.

Let us consider all the elements $\lambda \in k$ which satisfy $\lambda\lambda^* = 1$ and $\lambda^n = 1$, where, n is the dimension of M. These elements form a group and being a finite subgroup of the multiplicative group of invertible elements of the field k it is a cyclic group (see for instance [1, Thm. 17]). Let ν be a generator of the group, since $\nu^n = 1$ the order of ν is a divisor n' of n and the (see for instance [1, Thm. 17]). Let ν be a generator of the any element of Z is a square in Z, and this is certainly the case if n is odd. Therefore when n is odd the involutions of $P\,U^+(k, f)$ are $(p, n-p)$ cosets with p odd and $n-p$ even and we can apply the method of Mackey-Rickart to characterize the $(1, n-1)$ cosets among the $(p, n-p)$ cosets as being the only involutions of $\overline{T}$ of $P\,U^+(k, f)$ such that the double centralizer of $\{\overline{T}, \overline{R}\}$ in $P\,U^+(k, f)$ contains 4 involutions, where R is any involution commuting with $\overline{T}$, $\overline{R} \neq 1_L$, $\overline{T}$. This characterization permits us to extend [2, Thm. 28] to any odd dimensional space over a field of characteristic $\neq 2$.

From now on we assume $\dim M = n = 2m$. Then the order of ν is even, for if it were odd, say q, $-\nu$ would have order $2q$ which is also a divisor of $2m = n$ and $(-\nu)(-\nu)^* = 1$, showing that ν can not be a generator of Z. Now we need consider two different cases.

Case a. There exist an element $\varepsilon \in k$ such that $\varepsilon\varepsilon^* = 1$ and $\varepsilon^n = -1$. Without loss of generality we can assume $\varepsilon^2 = \nu$ since $\varepsilon^2 = \nu^{2h+1}$, for $\varepsilon^2 = \nu^{2h}$ implies $\varepsilon = \pm\nu^h$ and hence

$$\varepsilon^n = (\pm\nu^h)^n = 1,$$

and taking $\varepsilon' = \varepsilon\nu^{-h}$ we get $\varepsilon'^2 = \nu$, $\varepsilon'\varepsilon'^* = 1$ and $\varepsilon'^n = 1$. If the element S of the involutory coset $\overline{S}$ is such that $S^2 \in Z$ is not the square of an element of Z, $S^2 = \nu_L^{2s+1}$ and hence $S\varepsilon^{-(2s+1)}$ satisfies $(S\varepsilon^{-(2s+1)})^2 = 1_L$, that is, is a unitarian $(p, n-p)$ involution with $n-p$ odd since it has determinant -1. Although $S\varepsilon^{-(2s+1)} \notin U^+(k,f)$ we will still call $\overline{S}$ a $(p, n-p)$ coset and will say that $P\,U^+(k,f)$ contains $(p, n-p)$ cosets with $0 \leqslant p \leqslant n$, moreover any involution of $P\,U^+(k,f)$ is a $(p, n-p)$ coset.

Case *b*. There does not exist an element ε such that $\varepsilon\varepsilon^* = 1$, $\varepsilon^n = -1$. Then if $\overline{T}$ is an involution of $P\,U^+(k,f)$ and $T^2 = \nu^r$ is not a square in Z, $r = 2h+1$ and $(T\nu_L^{-r})^2 = \nu_L$. Hence we have that the involutions of $P\,U^+(k,f)$ are either $(p, n-p)$ cosets where p and $n-p$ are even or cosets defined by unitarian transformations T satisfying $T^2 = \nu_L$. In this case we call $\overline{T}$ a P-coset.

Let us study the involutions T, such that $T^2 = \nu_L$. Let x_1 be a non-isotropic vector and

$$x_1 T = \alpha_1 x_1 + y_1,$$

where $f(x_1, y_1) = 0$, and let us show that $f(y_1, y_1) \neq 0$. For if $f(y_1, y_1) = 0$, since

$$f(x_1, x_1) = f(x_1 T, x_1 T) = f(\alpha_1 x_1 + y_1, \alpha_1 x_1 + y_1) =$$
$$= \alpha_1 \alpha_1^* f(x_1, x_1)$$

we get $\alpha_1 \alpha_1^* = 1$, and from

$$\nu x_1 = x_1 T^2 = \alpha_1 (\alpha_1 x + y_1) + y_1 T,$$

that is,

$$y_1 T = (\nu - a_1^2)\, x_1 - \alpha_1 y_1$$

we deduce

$$0 = f(y_1, y_1) = f(y_1 T, y_1 T) = (\nu - \alpha_1^2)(\nu - \alpha_1^2)^* f(x_1, x_1)$$

which implies $\nu = \alpha_1^2$. This a contradiction for then

$$\alpha_1^{2m} = \nu^m = \pm 1,$$

and if $\alpha_1^{2m} = 1$, ν cannot be the generator of the group of ele-

ments satisfying $\lambda\lambda^* = 1$ and $\lambda^n = \lambda^{2m} = 1$ and if $\alpha_1^{2m} = -1$ we will be in case *a*.

Thus $f(y_1, y_1) \neq 0$ and x_1, y_1 span a non-isotropic plane P_1, invariant under T and, therefore the space $P_1^\perp$ orthogonal complement of P_1 is also invariant. We can choose a non-isotropic vector x_2 in $P_1^\perp$ such that $x_2 T = \alpha_2 x_2 + y_2$ where $f(x_2, y_2) = 0$, then the plane P_2 spanned by x_2 and y_2 is non-isotropic. Proceeding in this way we can decompose

$$M = P_1 \oplus P_2 \oplus \dots \oplus P_m$$

in a direct sum of non-isotropic planes invariant under T. The determinant of the restriction of T to P_i is

$$\begin{vmatrix} \alpha_i & 1 \\ \nu - \alpha_i^2 & -\alpha_i \end{vmatrix} = -\nu.$$

Hence the determinant of T is $(-\nu)^m$ which must be 1.

It is always possible to choose the non-isotropic vector x_1 such that $x_1 T = \alpha_1 x_1 + y_1$ where $f(x_1, y_1) = 0$ and $\alpha_1 \neq 0$, for if given a non-isotropic vector x we get $x T = \alpha x + y$ where $f(x, y) = 0$ and $\alpha \neq 0$ we take $x_1 = x$, if $\alpha = 0$, then

$$x T = y, \; y T = \nu x, \; f(x, x) = f(y, y).$$

Take $x + \beta y$, then

$$f(x + \beta y, x + \beta y) = (1 + \beta\beta^*) f(x, x) \neq 0$$

if $\beta\beta^* \neq -1$ and

$$f(x + \beta y, (x + \beta y) T) = f(x + \beta y, y + \beta\nu x) =$$
$$(\beta^* \nu^* + \beta) f(x, x) \neq 0$$

if $\nu^*\beta^* + \beta \neq 0$. To satisfy these conditions we can take $\beta = 1$ when $\nu \neq -1$, and when $\nu = -1$ any element $\gamma \notin k_0$ such that $\gamma\gamma^* \neq -1$ will do. With this β we have that $x_1 = x + \beta y$ is non-isotropic and $x_1 T = \alpha_1 x_1 + y_1$ with $\alpha_1 \neq 0$.

We show now that if x is any non-isotropic vector and $x\mathrm{T} = \alpha x + y$, where $f(x, y) = 0$, then $\alpha = h\alpha_1$ with $h \in k_0$. First notice that since

$$f(x, x) = f(x\mathrm{T}, x\mathrm{T}) = f(\alpha x + y, \alpha x + y) =$$
$$= \alpha\alpha^* f(x, x) + f(y, y), f(y, y) = (1 - \alpha\alpha^*) f(x, x),$$

now

$$0 = f(y, x) = f(y\mathrm{T}, x\mathrm{T}) = f((\nu - \alpha^2) x - \alpha y, \alpha x + y) =$$
$$= ((\nu - \alpha^2)\alpha^* - \alpha(1 - \alpha\alpha^*)) f(x, x)$$

implies $\nu\alpha^* = \alpha$. In particular $\nu\alpha_1^* = \alpha_1$, so by division we get

$$\alpha\alpha_1^{-1} = \alpha^*(\alpha_1^*)^{-1},$$

that is, $\alpha\alpha_1^{-1} = h \in k_0$. Notice that $\nu\alpha\alpha^* = \alpha^2$ also implies that $\alpha\alpha^*$ can not be a square a^2 in k_0 otherwise ν would be the square of the element αa^{-1} and $(\alpha a^{-1})(\alpha a^{-1})^* = 1$.

Let us take the transformation $\mathrm{S} = (c\alpha_1\theta)_L + d_L\mathrm{T}$, and let x be any non-isotropic vector, then

$$f(x((c\alpha_1\theta)_L + d_L\mathrm{T}), x((c\alpha_1\theta)_L + d_L\mathrm{T})) =$$
$$= (-c^2 r\alpha_1\alpha_1^* + d^2) f(x, x),$$

since $f(x, x\mathrm{T}) = h\alpha_1^*$. Thus if

$$d^2 - c^2 r\alpha_1\alpha_1^* = 1, \; f(x, x) = f(x\mathrm{S}, x\mathrm{S})$$

for every non-isotropic vector and this is a sufficient condition for S to be a unitarian transformation.

From now on we are going to assume that k_0 *has more than 5 elements.* Then there always exist elements c, d both different from zero such that $d^2 - c^2 r\alpha_1\alpha_1^* = 1$. This is immediate if $r\alpha_1\alpha_1^*$ is a square in k_0 (this is the case if k_0 is a finite fied) and if $r\alpha_1\alpha_1^*$ is not a square $d^2 - c^2 r\alpha_1\alpha_1^*$ is the norm of the elements of the form $d + c\gamma$ in the quadratic extension $k_0(\gamma)$ where

$\gamma^2 = r\alpha_1\alpha_1^*$ and there always exist elements of norm 1 with c and d different from zero.

As to the determinant of S, we have that the restriction of S to the plane P_i is

$$\begin{vmatrix} c\alpha_1\theta + \alpha_i d & d \\ d(\nu - \alpha_i^2) & c\alpha_1\theta - \alpha_i a \end{vmatrix} = -\nu(d^2 - c^2\alpha_1\alpha_1^* r) = -\nu.$$

Since $\nu\alpha_1^* = \alpha_1$. Hence $\det \mathrm{S} = (-\nu)^m \det \mathrm{T} = 1$.

3. When we try to apply the method of Mackey-Rickart to the characterization of $(1, n-1)$ cosets in case a and of $(2, n-2)$ cosets in case b we get into trouble since we can have 2 cosets $\bar{\mathrm{R}}$ and $\bar{\mathrm{S}}$ which commute but $\mathrm{RS} \neq \mathrm{SR}$. Hence the method has to be modified so that it will apply to the case of an even dimensional space M.

Now if $\bar{\mathrm{R}}$ is an involution and $\bar{\mathrm{S}}$ is a coset commuting with $\bar{\mathrm{R}}$ then

(1) $$\mathrm{RS} = \nu_L^r \mathrm{SR},$$

but since $\mathrm{R}^2 \in \mathrm{Z}$, $\mathrm{R}^2\mathrm{S} = \mathrm{SR}^2$; on the other hand (1) implies

$$\mathrm{R}^2\mathrm{S} = \nu_L^r \mathrm{RSR} = \nu_L^{2r}\mathrm{SR}^2,$$

so that $\nu^r = \pm 1$, that is, R and S either commute or anticommute. We will say that R is an irregular involution if there exist elements $\mathrm{S} \in U^*(k, f)$ such that $\mathrm{RS} = -\mathrm{SR}$, otherwise we say that $\bar{\mathrm{R}}$ is regular. It is clear that the $(p, n-p)$ cosets with $p \neq n - p$ are regular involutions and the other possible involutions might be regular or irregular. In particular if k is finite any (m, m) coset is an irregular involution, since then any non-isotropic subspace of M has an orthonormal basis.

Let us remark that since we assume that k_0 has more than 5 elements there always exist elements δ such that $\delta\delta^* = 1$ and $\delta^2 \neq \pm 1$.

LEMMA 2. An automorphism of $PU^*(k, f)$ cannot take a regular $(p, n-p)$ coset into an irregular involution.

PROOF. We only have to prove that the regular $(p, n-p)$ cosets have group-theoretical properties different from the properties of the irregular involutions.

If $\overline{T}$ is an irregular (m, m) coset and $\delta\delta^* = 1$, $\delta^2 \neq \pm 1$, the double centralizer of $\overline{T}$ does not contain the transformation which induce on the subspace M^+ the transformation δ_L and of M^- the transformation δ_L^{-1} but such transformation belongs to

$$C_{PU^+}\left((C_{PU^+}(\overline{T}))^2\right),$$

where $(C_{PU^+}(\overline{T}))^2$ denotes the group generated by the squares of the elements of $C_{PU^+}(\overline{T})$, for if

$$S \in (C_{PU^+}(\overline{T}))^2, \; S\,T = TS.$$

In case b there might also exist irregular P-involutions. Then the coset of the unitarian transformation

$$S = (c\,\alpha_1\,0)_L + d_L\,T,$$

where c, $d \neq 0$, does not belong to the double centralizer of $\overline{T}$ but it belongs to

$$C_{PU^+}\left((C_{P^+}((\overline{T})^2)\right).$$

Therefore if $\overline{T}$ is the coset of any irregular involution

$$C_{PU^+}\left(C_{PU^+}(\overline{T})\right) \neq C_{PU^+}\left((C_{P^+}(\overline{T}))^2\right).$$

On the other hand if T is a regular $(p, n-p)$ coset

$$C_{PU^+}\left((C_{P^+}(\overline{T}))^2\right)$$

consists of the cosets of the unitarian transformations of determinant 1 which induce scalar multiplication γ_L and γ_L' on the plus- and minus-spaces of the involution T. For, if $x_1 \in M^+$,

$x_2 \in M^-$ are two non-isotropic vectors and R is the transformation such that

$$x_1\,R = \delta^2\,x_1, \quad x_2\,R = \delta^{-2}\,x_2$$

and $x\,R = x$ for any vector x in the orthogonal complement of the plane spanned by x_1 and x_2, $\overline{R} \in (C_{PU^+}(\overline{T}))^2$. This shows that

$$C_{PU^+}\left((C_{P^+}(\overline{T}))^2\right) = C_{PU^+}\left(C_{P^+}(\overline{T})\right)$$

and establishes the lemma.

LEMMA 3. If $\dim M = n = 2m > 4$, in case a an automorphism of $PU^*(k, f)$ takes a $(1, n-1)$ coset into a $(1, n-1)$ coset.

PROOF. If $\dim M > 4$ and $\overline{T}$ is a $(1, n-1)$ coset, $\overline{T}$ is regular and given any other regular involution $\overline{R}$, $\overline{R} \neq \overline{T}$, 1_L, such that $\overline{R}\,\overline{T} = \overline{T}\,\overline{R}$, there exist only 4 involutions in the double centralizer of $\{\overline{R}, \overline{T}\}$. On the other hand if $\overline{T}$ is a $(p, n-p)$ coset, p, $n-p > 1$, and M^+ and M^- are the plus- and minus-spaces of the involution we define U as the unitarian involution whose plus-space is the plane spanned by two non-isotropic vectors $x_1 \in M^+$ and $x_2 \in M^-$; then $T\,U = U\,T$ and since $\dim M > 4$, $\overline{U}$ is a regular coset. Hence

$$C_{PU^+}\left(C_{PU^+}(\overline{T}, \overline{U})\right)$$

contains 8 involutions.

Since we have characterized the $(1, n-1)$ coset we can prove that the assertion of [2, Theorem 28] holds for case a.

LEMMA 4. In case b, if $\dim M = n = 2m > 4$, an automorphism of $PU^+(k, f)$ cannot take a $(2, n-2)$ coset into a regular $(p, n-p)$ coset with p and $n-p$ greater than 2.

PROOF. Let T be a unitarian $(2, n-2)$ involution it is clear that if $\overline{R}$ is a regular $(p, n-p)$ coset such that $\overline{T}\,\overline{R} = \overline{R}\,\overline{T}$ there are only 4 involutions in the double centralizer of $\{\overline{T}, \overline{R}\}$ in $P\,U^+(k, f)$. Let us assume then that $\overline{R}$ is a regular P-coset commuting with $\overline{T}$ and that the double centralizer of $\{\overline{T}, \overline{R}\}$ in $P\,U^+(k, f)$ contains more than 4 involutions, that is, it contains an involution $\overline{S}$ different from $\overline{I}_L$, $\overline{T}$, $\overline{R}$ and $\overline{T}\,\overline{R}$. Then this double centralizer contains a $(p, n-p)$ coset $\overline{U}$ different from $\overline{T}$, for either $\overline{S}$ in a $(p, n-p)$ coset and we take $\overline{U} = \overline{S}$ or $\overline{S}$ in a P-coset in which case $\overline{U} = \overline{R}\,\overline{S} \neq \overline{T}$ is a $(p, n-p)$ coset and belongs to the double centralizer. Let U^+ and U^- be the plus- and minus-spaces of one of the two unitarian involutions in $\overline{U}$; then $U^+R = U^+$, $U^-R = U^-$ and without loss of generality we can assume that if M^+ is the 2 dimensional plus-space of $T\,M^+$ is a proper subspace of U^+. Let x_1 be a non-isotropic vector such that $x_1 \in U^+$ and x_1 orthogonal to M^+, and let x_2 be a non-isotropic vector of U^-. We choose a non-zero element $\gamma \in k$ such that $z = x_1 + \gamma x_2$ is a non-isotropic vector then the $(2, n-2)$ coset defined by the involution W whose plus-space is the plane P spanned by z and

$$zR = z_1 x_1 + y_1 + \gamma(\alpha_2 x_2 + y_2)$$

belongs to $C_{PU^+}(\overline{T}, \overline{R})$ and does not commute with $\overline{U}$ since

$$U^+ \cap P = U^- \cap P = 0,$$

but this implies that

$$\overline{U} \notin C_{PU^+}\left(C_{PU^+}(\overline{T}, \overline{R})\right)$$

which is a contradiction. Hence we have shown that if $\overline{T}$ is a $(2, n-2)$ coset and $\overline{U}$ any regular involution commuting with $\overline{T}$, the double centralizer of $\{\overline{T}, \overline{U}\}$ in $P\,U^+(k, f)$ contains only 4 involutions.

Now let $T \in U^+(k, f)$ be a $(p, n-p)$ involution with p and $n-p$ greater than 2, M^+ and M^- its plus- and minus-spaces, and such that $\overline{T}$ is a regular coset. We assume first that $\dim M \neq 8$. Then if we take two non-isotropic planes P_1 and P_2 in M^+ and M^- respectively and define U as the unitarian involution whose plus space is $P_1 + P_2$ we have that U is regular, and commutes with T. Moreover the double centralizer of $\{\overline{T}, \overline{U}\}$ in $P\,U^+(k, f)$ contains 8 involutions and therefore $\overline{T}$ cannot be the image of $(2, n-2)$ coset under an automorphism of $P\,U^+(k, f)$.

When $\dim M = 8$ we cannot be sure that there exists a regular $\overline{U}$ defined as above so that we will have to admit that $\overline{U}$ might be an irregular involution, but then there might exist unitarian transformations of determinant 1 which commute with T and anticommute with U in which case the double centralizer of $\{\overline{T}, U\}$ contains only 4 involutions. To get around this situation we take

$$C_{PU^+}\left((C_{PU^+}(\overline{T}, U))^2\right)$$

which contains 8 involutions and notice that when k is a finite field there is nothing to prove since all the (4,4) cosets are irregular. Hence to prove the lemma for $\dim M = 8$ it will be enough to show that if k is an infinite field and T is a (2,6) involution with M^+ and M^- as subspaces, given any involution $\overline{R}$ commuting with $\overline{T}$ the group

$$C_{PU^+}\left((C_{PU^+}(\overline{T}, \overline{R}))^2\right)$$

contains only 4 involutions. This is clear if $\overline{R}$ is a $(p, 8-p)$ coset and if $\overline{R}$ is a P-coset we notice first that we can find elements c and d such that

$$S = (c\,\alpha\,0)_L + d_L\,R \in U^+(k, f)$$

and $S^2 = \gamma_L + \gamma'_L\,R$ with γ and γ' different from zero, so that any element of $U^+(k, f)$ which defines a coset belonging to the group

$$C_{PU^+}\left(C_{PU^+}(\overline{T}, \overline{R})^2\right)$$

commutes with R. Then we can see as before that if this group contains more than 4 involutions it contains a $(p, n-p)$ coset defined by a unitarian involution U. To get a contradiction as above it is enough to define W as the transformation which induces on the plane P the transformation δ_L^2 on M^+ δ_L^{-2} and the identity on the orthogonal complement of $M^+ + P$, for $\overline{W}\,U \neq U\,\overline{W}$ and

$$W \in \left(C_{PU^+}(\overline{T}, \overline{R})^2\right).$$

LEMMA 5. In case b and when $\dim M > 4$ any automorphism φ of $P\,U^+(k, f)$ takes a $(2, n-2)$ coset into a $(2, n-2)$ coset.

PROOF. So far we have seen that under an automorphism of $P\,U^+(k, f)$ a $(2, n-2)$ coset cannot be taken into an irregular involution or into a $(p, n-p)$ coset with p and $n-p$ greater than 2. Hence the image of a $(2, n-2)$ coset under an automorphism is either a $(2, n-2)$ coset or a regular P-coset. Assume that there exists a $(2, n-2)$ coset which is taken by φ into a P-coset. If the plane N of the $(2, n-2)$ coset is spanned by the non-isotropic vectors x_1 and x_2, let $x_3, x_4, \ldots, x_n$ be an orthogonal basis of the orthogonal complement of N. The $(2, n-2)$ coset defined by the planes N_{ij} spanned by x_i and x_j will be taken into $(2, n-2)$ cosets or regular P-cosets. By reordering the indices, if necessary, we can assume that the $(2, n-2)$ coset R_{1i} defined by the plane N_{1i} is taken by φ into a P-coset for $2 \leqslant i \leqslant r$ and into a $(2, n-2)$ coset if $r < i < n$. Since the product of $\overline{R}_{12}\,\overline{R}_{1j} = \overline{R}_{2j}$ is a $(2, n-2)$ coset its image under φ is a $(2, n-2)$ coset for $2 < j \leqslant r$ because

$$\varphi(\overline{R}_{2j}) = \varphi(\overline{R}_{12})\,\varphi(\overline{R}_{1j})$$

cannot be a P-coset. Therefore the $r-2$ $(2, n-2)$ cosets $\varphi(R_{2j})$ for $2 < j \leqslant r$ and the $n-r$ $(2, n-2)$ cosets $\varphi(R_{1j})$ for $j > r$ give $n-2$ $(2, n-2)$ cosets commuting with each other and commuting with the P-coset $\varphi(\overline{R}_{12})$, but the maximum number of $(2, n-2)$ cosets commuting with each other and with a P-coset is $\frac{1}{2}\,n = m$ since the planes corresponding to the $(2, n-2)$ cosets have to be orthogonal to each other. Thus we must have $n-2 = 2m-2 \leqslant m$, that is $m \leqslant 2$ which contradicts the assumption $\dim M = 2m > 4$. Hence the lemma is established and we can use Dieudonné's arguments to prove theorem 28 for case b.

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