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A Generalization of Z-groups

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A finite group with cyclic Sylow subgroups is called a Z-group. In his book on the theory of groups Zassenhaus described the structure of Z-groups. The aim of this paper is to give a generalization of this theorem. Roughly speaking, we consider solvable groups whose Sylow subgroups are of small class and are generated by few elements.

It is well known that, if the prime powers that divide the order of a finite group satisfy certain arithmetic conditions, the group is nilpotent. Our theorem gives also a generalization of this result.

In Section 1, besides explaining part of our notation, we give as a proposition a basic result that follows directly from definitions.

In Section 2, we establish the connection between the derived length of a factor group of G and the subindex of the first subgroup in the derived series of G that has a normal p-complement. Then, sufficient conditions are given for a solvable group to have a nilpotent commutator group.

Finally, in Section 3, we prove the Main Theorem.

1

All groups considered are finite. Let $| G |$ denote the order of the group G and $\pi(G)$ the set of primes that divide $| G |$.

If p is a prime number, a p-group is a group of order p^n and G is a p'-group if $p \notin \pi(G)$. More generally, if π is a set of primes and π' is its complementary, G is a π-group (π'-group) if $\pi(G) \subseteq \pi$ ($\pi(G) \subseteq \pi'$).

The largest normal p-subgroup (p'-subgroup) of G is denoted by $O_p(G)$ ($O_{p'}(G)$), and the inverse images in G of $O_{p'}(G/O_p(G))$ and $O_p(G/O_{p'}(G))$ are denoted by $O_{pp'}(G)$ and $O_{p'p}(G)$, respectively. The latter is the largest normal subgroup of G that has a normal p-complement.

The Fitting subgroup $F(G)$ is the largest normal nilpotent subgroup of G.

PROPOSITION 1.1. *$F(G) = \bigcap O_{p'p}(G)$, where p runs over the primes in $\pi(G)$.*

Proof. A nilpotent group has a normal p-complement for every prime p. Hence, $F(G) \subseteq O_{p'p}(G)$.

Since any subgroup of a group with a normal p-complement has a normal p-complement, $\bigcap O_{p'p}(G)$ is a normal nilpotent subgroup, so it is contained in $F(G)$.

As usual, we denote by $Z_i(G)$ the elements of the upper central series of G. That is, $Z_0(G)$ is the trivial group, $Z_1(G) = Z(G)$ is the center of G and $Z_i(G)/Z_{i-1}(G)$ is the center of $G/Z_{i-1}(G)$.

Let us recall that a finite group is p-solvable if each of its composition factors is either a p-group or a p'-group. A p-solvable group G has p-length 1 if $G/O_{p'p}(G)$ is a p'-group and $p \in \pi(G)$. A p'-group has p-length 0. The p-length of G is denoted by $l_p(G)$.

2

Let $D_0(G) \supseteq D_1(G) \supseteq D_2(G) \supseteq \cdots$ be the derived series of G. That is, $D_0(G) = G$, $D_1(G) = [G, G]$, and $D_{i+1}(G) = [D_i(G), D_i(G)]$. If G is solvable, $d(G)$ denotes its derived length.

It follows from the definitions that for any subgroup $N \lhd G$, $D_s(G/N) \simeq D_s(G) \cdot N/N$ holds for any s.

PROPOSITION 2.1. *$D_s(G)$ has a normal p-complement if and only if $s \geqslant d(G/O_{p'p}(G))$.*

Proof. Since $O_{p'p}(G)$ is the largest normal subgroup that has a normal p-complement, $D_s(G)$ will have a normal p-complement if and only if it is contained in $O_{p'p}(G)$. Equivalently, $D_s(G/O_{p'p}(G)) = 1$.

COROLLARY 2.1. *Let $| G | = p_1^{r_1} \cdots p_t^{r_t}$ and let $d_i = d(G/O_{p'_i p_i}(G))$. Then, $D_s(G)$ is nilpotent if and only if $s \geqslant \max(d_1, d_2, \ldots, d_t)$.*

Proof. $D_s(G)$ nilpotent is equivalent to saying that it has a normal p-complement for every p that divides $| G |$.

COROLLARY 2.2. *G has a nilpotent commutator subgroup if and only if $G/O_{p'p}(G)$ is abelian for each $p \in \pi(G)$. In particular, if $[G, G]$ is nilpotent, $l_p(G) \leqslant 1$ for any p.*

Proof. For the first statement, take $s = 1$ in the previous corollary. Since $G/O_{p'p}(G)$ does not contain nontrivial normal p-subgroups, when it is abelian it has to be a p'-group.

THEOREM 2.1. *Let G be a solvable group of order $p_1^{r_1}p_2^{r_2}\cdots p_t^{r_t}$ with $l_p(G)=1$ for $p=p_1, p_2,\ldots, p_t$. Let $m_i = m(P_i)$ be the minimal number of generators of the Sylow p_i-subgroup P_i and let $p_1 < p_2 < \cdots < p_t$. If $m_1 < p_2$ and $m_i < p_1$ for $i = 2, 3,\ldots, t$, the commutator subgroup $[G, G]$ is nilpotent.*

Proof. By [3, Lemma 1.2.5], $G/O_{p'p}(G)$ is isomorphic to a subgroup H of the general linear group $GL(m, F_p)$, where $m = m(P)$ and F_p is the field with p elements. If follows from a theorem of Itô (see [1, Corollary 5.2]) that, if each prime $p \in \pi(H)$ is greater than m, H is abelian. Then, by Corollary 2.2, $[G, G]$ is nilpotent.

3

Let P be a p-group and A be a p'-group of automorphisms of P. Then, A induces automorphisms in $\bar{P} = P/[P, P]$ and only the identity of A induces the identity automorphism. We use A again to denote the group of induced automorphisms. Since $\bar{P}$ is abelian, it is well known that $\bar{P} = C_{\bar{P}}(A) \times [\bar{P}, A]$, where $[\bar{P}, A]$ is the subgroup generated by the set $[x^{-1}x^\alpha \mid x \in \bar{P}, \alpha \in A\}$ and $C_{\bar{P}}(A) = \{x \in \bar{P} \mid x^\alpha = x$ for all $\alpha \in A\}$ (see [2, Theorem 5.2.3]).

Suppose that A is nontrivial and acts irreducibly on $P/\Phi(P)$, where $\Phi(P)$ denotes the Frattini subgroup of P. Then, it acts irreducibly on $\bar{P}/\Phi(\bar{P}) \cong P/\Phi(P)$ and since $C_{\bar{P}}(A) \neq \bar{P}$, $[\bar{P}, A] = \bar{P}$ must hold. But then, $[P, A] = P$, because $[\bar{P}, A]$ is the image of $[P, A]$ under the canonical epimorphism of P into $\bar{P}$.

In the following, $H(G)$ denotes the inverse image in $O_{p'p}(G)$ of the Frattini subgroup $\Phi(O_{p'p}(G)/O_{p'}(G))$ of the given factor group.

LEMMA 3.1. *Let G be a p-solvable group with $l_p(G)=1$. If $G/O_{p'p}(G)$ is nontrivial and acts irreducibly on $O_{p'p}(G)/H(G)$, then $[G, G]$ contains the Sylow p-subgroups of G.*

Proof. Let $G_0 = G/O_{p'}(G)$. Since $l_p(G) = 1$, $O_p(G_0) \cong P$, P a Sylow p-subgroup of G, and $O_{p'p}(G)/H(G) \cong P/\Phi(P)$. The assumption of the lemma implies that $G_0/O_p(G_0)$ acts irreducibly on $P/\Phi(P)$.

Let A be a complement of $O_p(G_0)$ in G_0. The preceding discussion implies that $O_p(G_0) = [O_p(G_0), A] \subseteq [G_0, G_0] \cong [G, G]\, O_{p'}(G)/O_{p'}(G)$. Therefore, $P \subseteq [G, G]$.

As before, $m(P)$ denotes the minimal number of generators of the group P.

THEOREM 3.1. *Let G be a solvable group, $\pi(G) = \{p_1,\ldots, p_t\}$ and let s_i be the smallest positive integer such that the greatest common divisor $(p_i^{s_i} - 1,$ $p_1p_2\cdots p_t) \neq 1$. Let P_i be a Sylow p_i-subgroup and assume that for each $p_i \in \pi(G)$, $l_G(p_i) = 1$ and $m(P_i) \leqslant s_i$. Then,*

(1) *G is the semidirect product of two of its Hall subgroups, $H_\rho \lhd G$ and $H_{\rho'}$.*

(2) *H_ρ and $H_{\rho'}$ are nilpotent. Moreover, $p_i \in \rho$ implies $m(P_i) = s_i$. In particular, if $s_i > m(P_i)$ for every i, G is nilpotent.*

(3) *$[G, G] = H_\rho \times [H_{\rho'}, H_{\rho'}]$.*

(4) *The order of the Fitting factor group $G/F(G)$ divides $\prod(p_i^{s_i} - 1)$, where the product runs through the $p_i \in \rho$. The exponent e of that group divides the least common multiple of the elements $p_i^{s_i} - 1$, where $p_i \in \rho$, and the greatest common divisor $(e, p_i^{s_i} - 1) \neq 1$.*

Proof. It follows from Fermat's little theorem and the definition of s_i that, if $i \neq j$, $s_i < p_j$. Hence, by Theorem 2.1, $[G, G]$ is nilpotent.

We show now that for each p_i, either the Sylow subgroup P_i is contained in $[G, G]$ and is normal in G, or G has a normal p_i-complement. For simplicity of notation, let $P_i = P$, $p_i = p$, $s_i = s$, and $m = m(P_i)$. Since the group $G/O_{p'p}(G)$ is isomorphic to a p'-subgroup of the linear group $GL(m, F_p)$, where F_p is the field with p elements, its order divides

$$N = (p^m - 1)(p^{m-1} - 1)\cdots(p - 1).$$

Now, if $m < s$, N is relatively prime to $p_1p_2\cdots p_t$ and hence, $G = O_{p'p}(G)$; that is, G has a normal p-complement. In particular, if $m(P_i) < s_i$ for each i, G is nilpotent.

Suppose then, that $m = s$. In this case, if G does not have a normal p-complement, $G/O_{p'p}(G)$ is a nontrivial p'-group whose order divides $p^s - 1$, but it is relatively prime to the product $(p^{s-1} - 1)\cdots(p - 1)$. Consequently, $G/O_{p'p}(G)$ acts irreducibly on $O_{p'p}(G)/H(G)$, and according to Lemma 3.1, $P \subset [G, G]$. Since the commutator subgroup is nilpotent, we get $P \lhd G$.

Next, we find the Hall subgroups referred to in (1). Let

$$\rho = \{p_i \in \pi(G) \mid P_i \subset [G, G]\},$$

then, the nilpotent Hall subgroup H_ρ is normal and G is the semidirect product of H_ρ and a complement, that is, a Hall ρ'-subgroup $H_{\rho'}$. Notice that, if G has a normal p-complement, $[P, P] = P \cap [G, G]$. Hence, $p \in \rho$ implies that $m(P) = s$.

Now, the subgroup $H_{\rho'}$, satisfies the conditions of the theorem. If it is not nilpotent, for some $p \in \rho'$, $O_{p'p}(H_{\rho'}) \neq H_{\rho'}$. That is, $H_{\rho'}$ does not have a normal p-complement and by the preceding argument, the Sylow subgroup P is contained in $[H_{\rho'}, H_{\rho'}] \subset [G, G]$. This contradicts the definition of ρ. Thus, $H_{\rho'}$, is nilpotent.

When $P \subset H_{\rho'}$, G has a normal p-complement and as noted above, $[P, P]$ is the Sylow p-group of the nilpotent group $[G, G]$. Therefore, $[G, G] = H_\rho \times [H_{\rho'}, H_{\rho'}]$.

It is clear that the Fitting subgroup of G is the direct product of H_ρ and the subgroup C of $H_{\rho'}$ which centralizes H_ρ. If $A(H_\rho)$ denotes the group of automorphisms of H_ρ obtained by conjugation which elements in $H_{\rho'}$, we have $A(H_\rho) \cong H_{\rho'}/C \cong G/F(G)$. The restriction of $A(H_\rho)$ to the Sylow subgroup P of H_ρ is nontrivial because $P \subset [G, G]$ and it defines a group whose order divides $p^s - 1$. Thus, $| G/F(G)|$ divides $\prod (p^s - 1)$, $p \in \rho$, and the order of any element of the factor group divides the least common multiple of the set of numbers $\{ p^s - 1 \mid p \in \rho\}$. If e denotes the exponent of $| G/F(G)|$, the greatest common divisor $(e, p^s - 1) \neq 1$, otherwise, $A(H)$ acts trivially on P.

It follows from [3, Theorem B and Lemma 3.2.2] that in a finite solvable group G whose Sylow p-subgroups have class less than p for each $p \in \pi(G)$, $l_p(G) = 1$.

Thus, if G is a solvable group such that, for every $p \in \pi(G)$, a Sylow p-subgroup P has class less than p and the greatest common divisor $(\prod_{i=1}^{s-1} (p^i - 1), | G |) = 1$, where $s = m(P)$, the structure of G is described by Theorem 3.1. In particular, we have the following well-known result about Z-group.

COROLLARY 3.1. *If all the Sylow subgroups of the finite group G are cyclic, G is the semidirect product of two cyclic Hall subgroups, $A = [G, G]$ and B. Moreover, if $A = \langle a \rangle$, $B = \langle b \rangle$ and $b^{-1}ab = a^r$, then $(r - 1, | A |) = 1$. (See* [6, *Chap. V, Theorem* 11]).

Proof. By a theorem of Burnside, G is solvable. Then, since it is clear that $l_p(G) = 1$, Theorem 3.1 applies. Since a nilpotent group with cyclic Sylow subgroups is itself cyclic, the first statement of the corollary follows from (1)–(3). Because $G = \langle a, b \rangle$, its commutator subgroup $A = \langle a \rangle$ is generated by $[a, b] = a^{r-1}$. Hence, $(r - 1, | A |) = 1$.

COROLLARY 3.2. *Let G be a finite group and let $r = m(P)$, where P is a Sylow p-subgroup. If for every $p \in \pi(G)$, $\prod_{i=1}^{r} (p^i - 1)$ and $| G |$ are relatively prime, G is nilpotent.*

Proof. Assume that the statement is false and let G be a minimal counter-example. Then, G is an odd order group and hence, solvable.

Suppose that G contains two distinct minimal normal subgroups M and N. Then, G is isomorphic to a subgroup of $G/M \times G/N$, and it is nilpotent. Therefore, G must have a unique minimal normal subgroup M. Since M is a p-group and G/M is nilpotent, G has a normal Sylow p-subgroup, and for every prime q, $q \neq p$, it has a normal q-complement. Therefore, $l_q(G) = 1$ for every prime in $\pi(G)$ and by part (2) of the theorem, G is nilpotent.

This corollary is a generalization of the nontrivial part of [5, Theorem 1], the theorem appears as problem 13 in [4, p. 285].

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